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RATIONAL APPROACH TO ANISOTROPY OF SAND

WEI WU

Lahmeyer International Ltd, Lyoner Str. 22 60528 Frankfurt, Germany

SUMMARY

The paper presents a constitutive model for the three-dimensional deformation–strength behaviour of inherently anisotropic sand. Based on non-linear tensorial functions, the model is developed without recourse to the concepts in plasticity theory such as yield surface and plastic potential. Benefited from the fact that no decomposition of strain into elastic and plastic parts is assumed, a unified treatment of anisotropic behaviour of deformation and strength is achieved. Anisotropy is characterized by a vector normal to the bedding plane. The extension of the constitutive model is furnished by incorporating the vector under consideration of the principle of objectivity and the condition of material symmetry. Distinct features of the model are its elegant formulation and its simple structure involving few material parameters. Model performance and comparison with experiments show that the model is capable of capturing the salient behaviour of anisotropic sand. © 1998 John Wiley & Sons, Ltd.

Key words: anisotropic sand; constitutive model; non-linear tensorial junctions

1. INTRODUCTION

Natural soils are often deposited in horizontal layers and then subjected to anisotropic stresses leading to preferred orientation of the particles. As a consequence, most natural sand deposits possess an inherently anisotropic structure which causes variation in deformation–strength characteristics as the loading direction changes. Conceptually distinction may be made between inherent and induced anisotropy with the former being caused by deposition and the latter being induced by stressing or straining. While this distinction may make sense for laboratory prepared specimens, for natural soils, however, it is often difficult to differentiate between inherent and induced anisotropy, since natural soils usually experience both inherent and induced anisotropy.

The mechanical behaviour of anisotropic soils is a topic of current interest for both experimental and theoretical investigations. Extensive experimental studies of inherent anisotropy have been made on laboratory prepared sand specimens[1–4]. These and other studies show that the deformation–strength characteristics depend highly on the orientation of the bedding plane with respect to the principal stress. The largest stiffness and strength occur for loading perpendicular to the bedding plane and the lowest values are observed for loading along the bedding plane. While the friction angle varies typically by about 10 per cent, changes of deformation modulus by a factor of two or more may be expected.

Correspondence to: Wei Wu, Lahmeyer International Ltd, Lyoner Str. 22 60528 Frankfurt, Germany

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As is often the case in geotechnical engineering, deformation and strength are traditionally dealt with in disjunct domains. Whereas deformation problems are usually solved based on elasticity theory, the studies on strength are almost exclusively within the domain of plasticity theory. For deformation problems involving anisotropy the well-established theory of elasticity can be readily resorted to. Since most deposited sands are transversely isotropic about an axis normal to the bedding plane, we are left with five material parameters to fully define the anisotropy in elasticity theory[5].

Theories to model anisotropic strength are developed mainly along the line of works for metals[6]. The basic idea is to modify the yield function by introducing an anisotropy tensor. The product of the anisotropy tensor and the stress tensor gives rise to joint invariants which can be used in formulating the yield function[7–10]. While the plastic constitutive models have achieved certain success, some drawbacks also come to light. A major disadvantage resides in the separate treatment of deformation and strength, which is intrinsic to plasticity theory as a result of the decomposition of strain into elastic and plastic parts. With the advent of computer and numerical technique, however, it is compelling to have a consistent treatment of deformation and strength in a constitutive model. In principle, this may be achieved by combining anisotropic elasticity and plasticity theory. However, such constitutive models lead inevitably to rather complicated formulations involving numerous material parameters, which poses great hinderance for solving boundary value problems. An exception in the development of theories for anisotropic solids was due to Boehler and Sawczuk[11] by making use of linear tensorial functions. However, their work concerned with stress and strain rate and therefore is applicable only to fully developed plastic flow, namely stress states at failure with vanishing stress rate.

As an alternative to the prevailing plasticity theory, the hypoplastic constitutive model based on non-linear tensorial functions has been proved to be promising in reproducing the salient behaviour of soil[12–18]. The hypoplastic model is developed within the framework of rational mechanics and differs substantially from the plastic models in that the basic concepts underlying the plasticity theory such as failure surface and flow rule are abandoned. Nor is it necessary to decompose strain and strain rate into elastic and plastic parts to account for the irreversible behaviour.

Our previous works concerned mainly with inherently isotropic soils. In an effort to incorporate inherent anisotropy, the hypoplastic model turns out to be particularly appealing, since the theory renders a unified treatment of the anisotropic deformation–strength behaviour possible. In the present paper, the hypoplastic model is extended to account for transverse anisotropy. In doing so, we attempt to establish a fairly general theoretical framework to incorporate the essentials of anisotropy while still preserving the merits of the model namely its elegant formulation with few material parameters.

2. BACKGROUND

Although the general principles in rational mechanics provide a rigorous framework for constitutive equations, it is still felt that this framework is too large to be directly applied to specific constitutive equations, these being inevitably so given the variety and complexity of materials to be covered. It is not surprising that an unambiguous definition of inherent and induced anisotropy is still lacking, since material symmetry in rational mechanics is dealt with mainly for simple materials with stress as function of deformation gradient. In what follows, the definition of induced and inherent anisotropy is provided and the corresponding representation for the constitutive equations is recapitulated.

We start with the following constitutive equation of rate-type for inherently isotropic materials

$$\mathring{\mathbf{T}} = \mathbf{H}(\mathbf{T}, \mathbf{D}) \tag{1}$$

where $\mathring{\mathbf{T}}$, $\mathbf{T}$ and $\mathbf{D}$ are the Jaumann stress rate, the Cauchy stress and strain rate respectively. The Jaumann stress rate is defined as follows:

$$\mathring{\mathbf{T}} = \dot{\mathbf{T}} + \mathbf{TW} - \mathbf{WT} \tag{2}$$

where $\mathbf{W}$ is the spin tensor. The strain rate and spin tensors are related to the velocity gradient as follows

$$\mathbf{D} = \tfrac{1}{2}[(\nabla\dot{\mathbf{x}}) + (\nabla\dot{\mathbf{x}})^{\mathrm{T}}], \quad \mathbf{W} = \tfrac{1}{2}[(\nabla\dot{\mathbf{x}}) - (\nabla\dot{\mathbf{x}})^{\mathrm{T}}] \tag{3}$$

Throughout the paper, we will keep using the tensor notation. Unless stated otherwise, bold lower and upper case letters are used to denote vectors and tensors, respectively. A superposed dot implies material time differentiation. A superposition of the letter T and of the number -1 signify transposition and inversion, respectively. In complying with the sign convention in soil mechanics compressive stress, strain and their rates are taken to be positive.

Material properties are intrinsic properties and hence should be independent of the observer namely the frame of reference. This physical consideration gives rise to the principle of objectivity according to which material properties are invariant to observer transformations. For rotational transformations this may be expressed as

$$\mathbf{H}(\mathbf{QTQ}^{\mathrm{T}}, \mathbf{QDQ}^{\mathrm{T}}) = \mathbf{QH}(\mathbf{T}, \mathbf{D})\,\mathbf{Q}^{\mathrm{T}} \tag{4}$$

in which $\mathbf{Q}$ is an arbitrary orthogonal tensor. The principle of objectivity implies that the function $\mathbf{H}(\mathbf{T}, \mathbf{D})$ must be an isotropic function of $\mathbf{T}$ and $\mathbf{D}$. In the most general case, the representation theorem for an isotropic tensor-valued function of two symmetric tensors can be written as

$$\begin{aligned}\mathring{\mathbf{T}} = {} & \phi_0\mathbf{1} + \phi_1\mathbf{T} + \phi_2\mathbf{D} + \phi_3\mathbf{T}^2 + \phi_4\mathbf{D}^2 + \phi_5(\mathbf{TD} + \mathbf{DT}) \\ & + \phi_6(\mathbf{T}^2\mathbf{D} + \mathbf{DT}^2) + \phi_7(\mathbf{TD}^2 + \mathbf{D}^2\mathbf{T}) \ + \phi_8(\mathbf{T}^2\mathbf{D}^2 + \mathbf{D}^2\mathbf{T}^2)\end{aligned} \tag{5}$$

where $\mathbf{1}$ is the unit tensor. The coefficients in (5), ϕ_i $(i = 0, \ldots, 8)$, are functions of the invariants and joint invariants of $\mathbf{T}$ and $\mathbf{D}$

$$\phi_i = \phi_i(\mathrm{tr}\,\mathbf{T}, \mathrm{tr}\,\mathbf{T}^2, \mathrm{tr}\,\mathbf{T}^3, \mathrm{tr}\,\mathbf{D}, \mathrm{tr}\mathbf{D}^2, \mathrm{tr}\mathbf{D}^3, \mathrm{tr}(\mathbf{TD}), \mathrm{tr}(\mathbf{T}^2\mathbf{D}), \mathrm{tr}(\mathbf{TD}^2), \mathrm{tr}(\mathbf{T}^2\mathbf{D}^2)) \tag{6}$$

where tr represents the trace of a tensor.

If the material in concern possesses some symmetries as a result of structural ordering of the material points in a body, there exist particular changes of reference configuration which leave the stress rate resulting from an arbitrary strain rate invariant. The collection of such transformations is a measure of the degree of material symmetry and is known to build a group. The constitutive equation must be invariant under the group of symmetry transformations (isotropy group). Isotropic materials are known to possess the largest isotropy group, i.e. full orthogonal group. A material is said to be anisotropic, if its isotropy group is a subgroup of the full orthogonal group. Typical symmetry transformations include rotations about an axis (transverse anisotropy) and symmetry about planes (orthogonal anisotropy).

As a basis for developing the concepts underlying the symmetry properties, let us consider the effect of a change of reference configuration, say $\mathbf{Q}$, upon the strain rate and on the

constitutive equation. The strain rate will transform according to $\mathbf{QDQ}^{\mathrm{T}}$. Due to the dependence on stress, the responses of constitutive equation (1) are generally not the same in both reference configurations. The material as described by the model is therefore anisotropic. In order to classify anisotropy, let us consider such changes of reference configurations that leave the stress rate tensor unchanged

$$\mathbf{H}(\mathbf{T}, \mathbf{QDQ}^{\mathrm{T}}) = \mathbf{QH}(\mathbf{T}, \mathbf{D})\, \mathbf{Q}^{\mathrm{T}} \tag{7}$$

By way of example, we may consider the following stress states: a hydrostatic stress state (with three coincident eigenvalues), an axissymmetrical stress state (with two coincident eigenvalues) and an arbitrary stress state (with three distinct eigenvalues). For the above stress states constitutive equation (1) can be shown to be isotropic, transversely anisotropic and orthogonally anisotropic, respectively. Anisotropy is attained by the apparent dependence of the constitutive equation on stress and is therefore stress induced. By the way, the least degree of symmetry in such a constitutive equation is orthotropic, which is obtained for a stress tensor with three distinct eigenvalues.

The above condition of symmetry (7) bears some resemblance with the principle of objectivity (4), since in both cases the response of the constitutive equation under orthogonal transformations is considered. Nevertheless, the principle of objectivity holds for arbitrary orthogonal transformations, whereas the condition of symmetry is fulfilled only for particular transformations.

Constitutive equation (1) cannot account for inherent anisotropy since the response in a hydrostatic stress state is invariant under the full group of symmetry transformations. We seek to establish the framework to incorporate inherent anisotropy into constitutive equation (1). We set off by considering a material element of transverse anisotropy and referring to a co-ordinate system in which the axis x_1 coincides with the axis of isotropy $\mathbf{a}$ as shown in Figure 1. In planes normal to $\mathbf{a}$ the material is assumed to be isotropic. The principal stresses are defined by referring

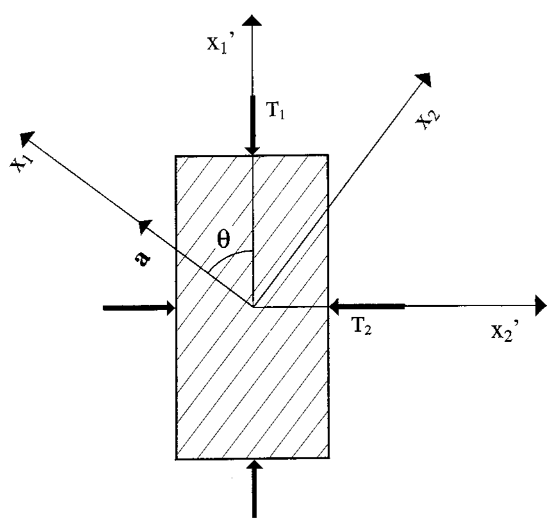

Figure 1. Coordinate system

to another co-ordinate system in which the axis x'_1 is aligned with the major principal stress. The angle between x_1 and x'_1, called the bedding angle, is denoted by θ. In order to incorporate inherent anisotropy, constitutive equation (1) should depend explicitly on the orientation of the bedding plane

$$\mathring{\mathbf{T}} = \mathbf{H}(\mathbf{T}, \mathbf{D}, \mathbf{a}) \tag{8}$$

The principle of objectivity requires that the function $\mathbf{H}(\mathbf{T}, \mathbf{D}, \mathbf{a})$ should be objective. That is

$$\mathbf{H}(\mathbf{QTQ}^{\mathrm{T}}, \mathbf{QDQ}^{\mathrm{T}}, \mathbf{Qa}) = \mathbf{QH}(\mathbf{T}, \mathbf{D}, \mathbf{a})\,\mathbf{Q}^{\mathrm{T}} \tag{9}$$

Although being an isotropic function of its arguments, the response of constitutive equation (8) with respect to strain rate is generally anisotropic. The degree of symmetry is characterized by the transformations under which $\mathbf{T}$ or $\mathbf{a}$ remains invariant.

Distinction between induced and inherent anisotropy may be made by the invariance of $\mathbf{T}$ or of $\mathbf{a}$ under the transformation $\mathbf{Q}$. Induced anisotropy of a material element is there if the orthogonal transformations $\mathbf{Q}$ applied to $\mathbf{D}$ and $\mathbf{a}$ leaving the stress rate invariant

$$\mathbf{H}(\mathbf{T}, \mathbf{QDQ}^{\mathrm{T}}, \mathbf{Qa}) = \mathbf{QH}(\mathbf{T}, \mathbf{D}, \mathbf{a})\,\mathbf{Q}^{\mathrm{T}} \tag{10}$$

form a subgroup of the full orthogonal group. Analogously, inherent anisotropy is there if the orthogonal transformations $\mathbf{Q}$ applied to $\mathbf{T}$ and $\mathbf{D}$ leaving the stress rate invariant

$$\mathbf{H}(\mathbf{QTQ}^{\mathrm{T}}, \mathbf{QDQ}^{\mathrm{T}}, \mathbf{a}) = \mathbf{QH}(\mathbf{T}, \mathbf{D}, \mathbf{a})\,\mathbf{Q}^{\mathrm{T}} \tag{11}$$

form a subgroup of the full orthogonal group.

Assuming that the above transformation holds for both proper and improper transformations, it can be shown that $\mathbf{H}(\mathbf{T}, \mathbf{D}, \mathbf{a})$ must be an even function of $\mathbf{a}$. It is clear $\mathbf{a}$ is indiscriminable from $-$ a, which means that the plane normal to $\mathbf{a}$ is a plane of symmetry.

Now, we will briefly recapitulate the representation theorem for isotropic tensorial functions of two symmetric second-order tensors and a vector. For functions of even order (scalar and tensor isotropic functions) the vector $\mathbf{a}$ in (8) can be replaced by the tensor variable $\mathbf{A} = \mathbf{a} \otimes \mathbf{a}$. With reference to the co-ordinate system x′ the tensor $\mathbf{A}$ can be written out as follows:

$$\mathbf{A} = \begin{bmatrix} +\cos\theta\cos\theta & -\sin\theta\cos\theta & 0 \\ -\sin\theta\cos\theta & +\sin\theta\sin\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{12}$$

Analogously to the representation theorem in (5) and (6), the representation of $\mathring{\mathbf{T}}$ as a tensorial function of $\mathbf{T}$, $\mathbf{D}$ and $\mathbf{A}$ can be readily written as[19]

$$\begin{aligned} \mathring{\mathbf{T}} = {} & \phi_0 \mathbf{1} + \phi_1 \mathbf{T} + \phi_2 \mathbf{D} + \phi_3 \mathbf{A} \\ & + \phi_4(\mathbf{TD} + \mathbf{DT}) + \phi_5(\mathbf{AT} + \mathbf{TA}) + \phi_6(\mathbf{AD} + \mathbf{DA}) \\ & + \phi_7 \mathbf{T}^2 + \phi_8 \mathbf{D}^2 + \phi_9(\mathbf{T}^2\mathbf{D} + \mathbf{DT}^2) + \phi_{10}(\mathbf{TD}^2 + \mathbf{D}^2\mathbf{T}) \\ & + \phi_{11}(\mathbf{AT}^2 + \mathbf{T}^2\mathbf{A}) + \phi_{12}(\mathbf{AD}^2 + \mathbf{D}^2\mathbf{A}) \end{aligned} \tag{13}$$

The coefficients in (13), $\phi_i (i = 0, \ldots, 12)$, are functions of the invariants and joint invariants of $\mathbf{T}$, $\mathbf{D}$ and $\mathbf{A}$

$$\phi_i = \phi_i(\mathrm{tr}\,\mathbf{T}, \mathrm{tr}\,\mathbf{T}^2, \mathrm{tr}\,\mathbf{T}^3, \mathrm{tr}\,\mathbf{D}, \mathrm{tr}\mathbf{D}^2, \mathrm{tr}\mathbf{D}^3, \mathrm{tr}(\mathbf{TD}), \mathrm{tr}(\mathbf{T}^2\mathbf{D}), \mathrm{tr}(\mathbf{TD}^2), \mathrm{tr}(\mathbf{T}^2\mathbf{D}^2),$$
$$\mathrm{tr}(\mathbf{AT}), \mathrm{tr}(\mathbf{AD}), \mathrm{tr}(\mathbf{AT}^2), \mathrm{tr}(\mathbf{AD}^2), \mathrm{tr}(\mathbf{ATD})) \tag{14}$$

In principle, anisotropic constitutive equations can be developed by combining the terms provided by (13) and (14). In view of the numerous possibilities of combinations, however, it can be a tedious exercise till a workable constitutive equation is found. The recent works by Wu and Bauer[16] and Wu *et al.*[18] showed the way from representation theorem to constitutive equations for inherently isotropic soils. In the present paper anisotropy is accounted for by extending the constitutive equation of inherent isotropy.

3. HYPOPLASTIC CONSTITUTIVE MODEL

We are concerned here with the following hypoplastic constitutive equation proposed by Wu and Kolymbas[13]:

$$\mathring{\mathbf{T}} = \mathbf{L}(\mathbf{T}, \mathbf{D}) + \mathbf{N}(\mathbf{T})\|\mathbf{D}\| \tag{15}$$

where $\mathbf{L}(\mathbf{T}, \mathbf{D})$ and $\mathbf{N}(\mathbf{T})$ are isotropic tensorial functions, and $\|\cdot\|$ stands for a norm and is defined by $\|\mathbf{D}\| = \sqrt{\mathrm{tr}\,\mathbf{D}^2}$. We refer to References 13, 16 and 18 for details of the inherently isotropic model.

Granted that the behaviour to be described is rate-independent constitutive equation (15) is necessarily positively homogeneous of the first degree in $\mathbf{D}$. That is

$$\mathbf{L}(\mathbf{T}, \lambda\mathbf{D}) + \mathbf{N}(\mathbf{T})\|\lambda\mathbf{D}\| = \lambda\mathbf{L}(\mathbf{T}, \mathbf{D}) + \lambda\mathbf{N}(\mathbf{T})\|\mathbf{D}\| \tag{16}$$

where λ is a positive but otherwise arbitrary scalar. Besides, the strength and deformation characteristics of granular materials are known to depend on the stress level. We assume a linear dependence of the tangential stiffness on the stress level, which implies the first degree homogeneity of constitutive equation (15) in stress

$$\mathbf{L}(\lambda\mathbf{T}, \mathbf{D}) + N(\lambda\mathbf{T})\|\mathbf{D}\| = \lambda\mathbf{L}(\mathbf{T}, \mathbf{D}) + \lambda N(\mathbf{T})\|\mathbf{D}\|, \tag{17}$$

where λ is an arbitrary scalar.

To show the difference between the theory of hypoplasticity and of plasticity in more detail, constitutive equation (15) can be recast in a more convenient form by virtue of Euler's theorem for homogeneous functions

$$\mathring{\mathbf{T}} = (\mathsf{L} + \mathbf{N} \otimes \mathsf{D})\,\mathbf{D}, \tag{18}$$

where $\mathsf{D} = \mathbf{D}/\|\mathbf{D}\|$ stands for the direction of strain rate, and the symbol $\otimes$ denotes an outer product between two tensors.

The two terms in the parenthesis in (18) represent the tangential stiffness tensor. It is apparent from (18) that the tangential stiffness tensor depends not only on stress but also on the direction of strain rate. As compared with elastoplasticity theory, hypoplastic constitutive models are incrementally non-linear. Note that the distinction between loading and unloading is of unimpor-

tance for the hypoplastic constitutive equation, since the non-linear part always works for both loading and unloading. As a matter of fact, the determination of the yield surface and hence of loading and unloading for granular materials is usually rather subjective, since the stress–strain curves generally do not show clear yielding point as many metallic materials.

Constitutive equation (15) can be regarded pro forma as the sum of a linear term and a non-linear term. For a given strain rate the linear and the non-linear term give rise to increase and decrease of stress rate, respectively. Therefore, the linear term may be regarded as the constructive term and the non-linear term as the destructive term. The effect of anisotropy can be thought of as a kind of weakening to an otherwise isotropic medium. It is then intuitively reasonable to incorporate the anisotropy tensor as an enhancement to the destructive term. In doing so, we will not directly operate the stress tensor and the anisotropy tensor. Instead, we seek to obtain a tensorial function $\mathbf{M}$ of the tensors $\mathbf{N}$ and $\mathbf{A}$

$$\mathring{\mathbf{T}} = \mathbf{L}(\mathbf{T}, \mathbf{D}) + \mathbf{M}(\mathbf{A}, \mathbf{N})\|\mathbf{D}\| \tag{19}$$

It follows that the tensorial function $\mathbf{M}(\mathbf{A}, \mathbf{N})$ must be symmetric and an isotropic function of its arguments. The former assertion stems from the fact that $\mathring{\mathbf{T}}$ is symmetric, while the latter is a direct consequence of objectivity. $\mathbf{M}(\mathbf{A}, \mathbf{N})$ may be obtained according to the representation theorem (13). We further assume that $\mathbf{M}(\mathbf{A}, \mathbf{N})$ is homogeneous of the first degree in $\mathbf{N}$. This is necessary to retain the constitutive equation as a whole homogeneous of the first degree in stress. To this end, we make use of Euler's theorem for homogeneous functions to recast $\mathbf{M}(\mathbf{A}, \mathbf{N})$ in (19) in a different form

$$\mathbf{M}(\mathbf{A}, \mathbf{N}) = \mathsf{B}(\mathbf{A})\,\mathbf{N} \tag{20}$$

where the forth-order tensor defined by $\mathsf{B}(\mathbf{A}) = \partial\mathbf{M}/\partial\mathbf{N}$ is a function of $\mathbf{A}$ and is independent of $\mathbf{N}$.

Since both $\mathbf{M}$ and $\mathbf{N}$ are symmetric tensors of the second order, the above expression may be rewritten in the matrix form by replacing B and $\mathbf{N}$ with matrices of 6×6 and 6×1 respectively. With reference to the coordinate system $\mathbf{x}$ in Figure 1, the matrices can be written out explicitly as follows:

$$\{\mathbf{M}\} = [\mathsf{B}]\{\mathbf{N}\} = \begin{bmatrix} b_{11} & b_{12} & b_{12} & 0 & 0 & 0 \\ b_{12} & b_{22} & b_{23} & 0 & 0 & 0 \\ b_{12} & b_{23} & b_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & b_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & b_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & b_{22}-b_{23} \end{bmatrix} \begin{pmatrix} N_{11} \\ N_{22} \\ N_{33} \\ N_{23} \\ N_{13} \\ N_{12} \end{pmatrix} \tag{21}$$

The elements in matrix $[\mathsf{B}]$ are the so-called coefficients of anisotropy, which remain to be determined. Note the rule of correspondence between the indices of the second-order tensor and the column matrix ($11\to 1$, $22\to 2$, $33\to 3$, $23\to 4$, $13\to 5$, $12\to 6$). The matrix $[\mathsf{B}]$ can be compared to the stiffness tensor in elasticity theory. As may be easily ascertained, matrix $[\mathsf{B}]$ contains five independent elements (coefficients of anisotropy) and is invariant under the transformations corresponding to rotation around $\mathbf{a}$. To further simplify the matter, we assume that the off-diagonal elements in (21) vanish. As will be shown thereafter, this assumption greatly simplifies the determination of the anisotropy coefficients and still remains the fundamental

features of transverse anisotropy. We are then left with three coefficients and denote them by $b_{11} = \alpha$, $b_{22} = \beta$ and $b_{44} = \gamma$. To this end, the matrix $[\mathsf{B}]$ simplifies to

$$\{\mathbf{M}\} = [\mathsf{B}]\{\mathbf{N}\} = \begin{bmatrix} \alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma & 0 & 0 \\ 0 & 0 & 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta \end{bmatrix} \begin{pmatrix} N_{11} \\ N_{22} \\ N_{33} \\ N_{23} \\ N_{13} \\ N_{12} \end{pmatrix} \tag{22}$$

It should be pointed out that the representations of matrix $[\mathsf{B}]$ in (21) and (22) are obtained with reference to the co-ordinate system $\mathbf{x}$ in Figure 1. If another co-ordinate system different from $\mathbf{x}$ is chosen, the matrix $[\mathsf{B}]$ must be transformed. In most engineering problems, the co-ordinate system is often chosen to be attached to the construction works. This co-ordinate system, e.g. $\mathbf{x}'$ in Figure 1, is generally different from $\mathbf{x}$. Under the transformation of co-ordinate system from $\mathbf{x}$ to $\mathbf{x}'$, the matrices $\mathbf{M}$ and $\mathbf{N}$ transform according to

$$\{\mathbf{M}'\} = [\mathsf{S}]\{\mathbf{M}\}, \qquad \{\mathbf{N}'\} = [\mathsf{S}]\{\mathbf{N}\} \tag{23}$$

where the matrix of transformation $[S]$ is given by[20]

$$[\mathsf{S}] = \begin{bmatrix} C^2 & S^2 & 0 & CS & 0 & 0 \\ S^2 & C^2 & 0 & -CS & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -2CS & 2CS & 0 & C^2 - S^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & C & -S \\ 0 & 0 & 0 & 0 & S & C \end{bmatrix} \tag{24}$$

with $C = \cos\theta$, $S = \sin\theta$ and $CS = \cos\theta\sin\theta$. Substituting (23) into (20) we have

$$[\mathsf{S}]\{\mathbf{M}\} = [\mathsf{B}][\mathsf{S}]\{\mathbf{N}\} \tag{25}$$

The transformed matrix $[\mathsf{B}']$ can be readily seen to be

$$[\mathsf{B}'] = [\mathsf{S}]^{\mathrm{T}}[\mathsf{B}][\mathsf{S}] \tag{26}$$

Note that the inversion of the orthogonal matrix $[\mathsf{S}]$ in the above expression is replaced by the transposition.

Alternatively, we may also work with tensorial functions instead of the matrices. Nevertheless, the matrix representation is to be preferred owing to its resemblance to the elasticity theory. It can be shown that the simplified expression in (22) corresponds to the following tensorial function[11]

$$\mathbf{M}(\mathbf{A}, \mathbf{N}) = (\alpha + \gamma - 2\beta)\,\mathrm{tr}(\mathbf{AN})\,\mathbf{A} + \gamma\mathbf{N} + (\beta - \gamma)(\mathbf{AN} + \mathbf{AN}) \tag{27}$$

It is easily seen that if the coefficients in (22) and (27) are equal to unity, i.e. $\alpha = 1$, $\beta = 1$ and $\gamma = 1$, we have $\mathbf{M}(\mathbf{A}, \mathbf{N}) = \mathbf{N}$ and the constitutive equation of inherent isotropy is recovered. In this

sense, the constitutive equation of inherent isotropy can be regarded as a special case of the extended model.

Having established the framework to incorporate inherent anisotropy, we proceed to consider the following specific constitutive equation proposed by Wu.[21]

$$\mathring{\mathbf{T}} = c_1(\operatorname{tr}\mathbf{T})\,\mathbf{D} + c_2\,\frac{\operatorname{tr}(\mathbf{TD})\mathbf{T}}{\operatorname{tr}\mathbf{T}} + \left(c_3\,\frac{\mathbf{T}^2}{\operatorname{tr}\mathbf{T}} + c_4\,\frac{\mathbf{T}_d^2}{\operatorname{tr}\mathbf{T}}\right)\|\mathbf{D}\|, \tag{28}$$

where $c_i\,(i = 1, \ldots, 4)$ are dimensionless constants. The deviatoric stress tensor in (28) is given by $\mathbf{T}_\mathrm{d} = \mathbf{T} - 1/3(\operatorname{tr}\mathbf{T})\mathbf{1}$.

The hypoplastic constitutive equation has been employed successfully to describe various aspects of the behaviour of granular materials. In particular, remarkable progress was achieved by Wu *et al.*[18] by integrating the critical state into (15). The constitutive model by Wu *et al.* covers a broad range of stress level and density and accounts for both initial deformation and fully developed flow.

4. MODEL PERFORMANCE

Since constitutive equation (28) is written in tensorial notation, the essential features of the model can be hardly perceived without detailing it in component form. In what follows, we consider several element tests to bring out the features of the constitutive equation.

4.1. Isotropic stressing and straining

We start by considering the simplest case of isotropic stressing and straining. By stressing and straining we mean that the tests are performed under stress control and strain control respectively. The boundary conditions can be specified for isotropic stressing by

$$\dot{T}_{ij} = \dot{p}\ \ (i = j), \qquad \dot{T}_{ij} = 0\ \ (i \neq j) \tag{29}$$

and for isotropic straining by

$$D_{ij} = \pm 1\ \ (i = j), \qquad D_{ij} = 0\ \ (i \neq j) \tag{30}$$

where $\dot{p} = \operatorname{tr}\dot{\mathbf{T}}$. Since the constitutive equation in concern is rate-independent, the strain rate can be chosen to be $+1$ for compression and -1 for extension.

We proceed to consider isotropic stressing with $T_{ij} = p\,(i = j)$, $T_{ij} = 0\ (i \neq j)$. To obtain the governing differential equations, we first insert the expressions $\mathbf{A}$ in (12) and $\mathbf{N}$ in (28) into (27) to get $\mathbf{M}(\mathbf{A}, \mathbf{N})$ and then replace the corresponding terms in the extended constitutive equation (19) by $\mathbf{L}(\mathbf{T}, \mathbf{D})$ in (28) and by the above derived $\mathbf{M}(\mathbf{A}, \mathbf{N})$. Due to symmetry of stress, strain and their rates, we are left with six equations of which the following four are non-trivial:

$$9\,\frac{\dot{T}_{11}}{p} = (c_2 + 9c_1)\,D_{11} + c_2 D_{22} + c_2 D_{33} + c_3(\alpha\cos^2\theta + \gamma\sin^2\theta)\,\|\mathbf{D}\| \tag{31}$$

$$9\,\frac{\dot{T}_{22}}{p} = c_2 D_{11} + (c_2 + 9c_1)\,D_{22} + c_2 D_{33} + c_3(\alpha\sin^2\theta + \gamma\cos^2\theta)\,\|\mathbf{D}\| \tag{32}$$

$$9\,\frac{\dot{T}_{33}}{p} = c_2 D_{11} + c_2 D_{22} + (c_2 + 9c_1)\, D_{33} + c_3 \gamma \|\mathbf{D}\| \tag{33}$$

$$9\,\frac{\dot{T}_{12}}{p} = c_1 D_{12} - c_3(\alpha - \gamma)\, \sin\theta \cos\theta \|\mathbf{D}\|, \tag{34}$$

where $\|\mathbf{D}\| = \sqrt{D_{11}^2 + D_{22}^2 + D_{33}^2 + 2D_{12}^2}$.

As can be seen from (31) and (32) that the two in-plane stress rates depend on the bedding angle. The resultant components of strain rate are generally not equal. As might be expected, the out-of-plane stress rate in (33) is independent of the bedding angle.

The last equation shows that the shear stress rate is different from zero and is also a function of the bedding angle. This means that the material element undergoes not only volume change but also distortion. It is interesting to observe that the coefficient β does not appear in the equations for isotropic stressing. The behaviour is controlled by the coefficients α and γ. For the case of inherent isotropy with $\alpha = 1$, $\beta = 1$, $\gamma = 1$, a single equation is obtained:

$$\frac{\dot{p}}{p} = 3c_1 \dot{\varepsilon} + c_2 \dot{\varepsilon} + \frac{c_3}{3} |\dot{\varepsilon}|, \tag{35}$$

where $\dot{\varepsilon} = D_{11} = D_{22} = D_{33}$.

In order to gain numerical results, the material parameters in constitutive equation (28) and the coefficients of anisotropy (27) must be determined. In the present paper, the following approach is adopted. The parameters c_i $(i = 1, \ldots, 4)$ are first identified for a hypothetical isotropic material. The following constants are used to obtain the numerical results throughout the paper: $c_1 = -66{\cdot}4$, $c_2 = -509{\cdot}4$, $c_3 = -503{\cdot}7$, $c_4 = 694{\cdot}5$. For details of the calibration procedure for the inherently isotropic constitutive model, the readers are referred to Reference 16. The coefficients of anisotropy can be then identified by fitting experimental results on anisotropic specimens. Unless stated otherwise, the following constants are used for numerical simulations: $\alpha = 0{\cdot}9$, $\beta = 1{\cdot}0$, $\gamma = 1{\cdot}1$. As will be shown thereafter, some fine-tuning of the coefficients is still necessary in fitting the experimental data.

With the parameters given above we are in a position to solve the equation system (31) to (33). The equation system of four simultaneous non-linear ODE may be solved under consideration of the boundary conditions (29) and of an initially hydrostatic stress state. We make use of the subroutines provided by Numerical Recipes.[22] The numerical results for a bedding angle of $\theta = 30^\circ$ are provided in Figure 2(a). It can be seen from Figure 2(a) that the three normal strain components are unequal with $E_{11} < E_{22} < E_{33}$. The shear component E_{12} is seen to evolve, however, at a lower rate than the normal components. This means an initially right cube will deform into a rhomb as depicted in Figure 3(a).

The problem of isotropic straining can be handled in similar manner. The governing equations are, however, different from their counterparts (31)–(34) in view of the fact that the stress components are unequal and the components of strain rate are equal. The underlying equations can be readily obtained and will not be given here. The solution of the ODE is more straightforward than that for isotropic stressing, since the components of strain rate are specified. For a given strain rate we need only to update the stress, which serves as the initial value for further calculations. The numerical results obtained for a bedding angle of $\theta = 30^\circ$ are presented in Figure 2(b), where the resultant stresses are depicted over strain. The three normal stress components differ slightly from one another, while the shear stress evolves more slowly than the

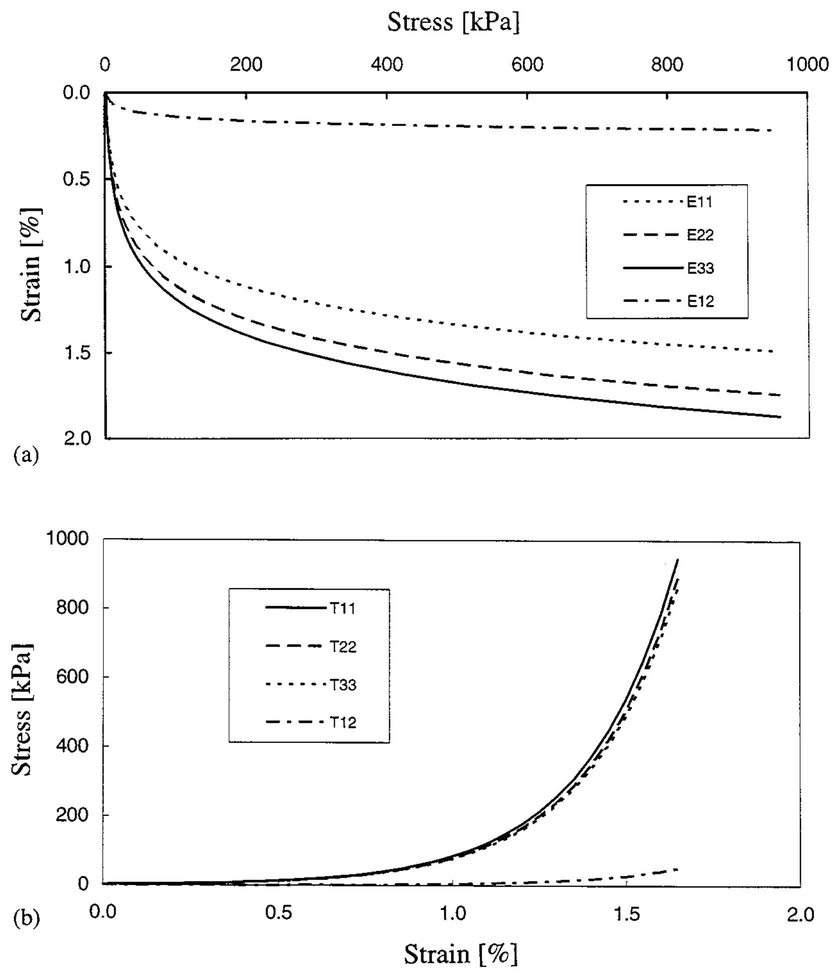

Figure 2. Simulation of isotropic compression: (a) isotropic stressing; (b) isotropic straining

normal stresses. A cube under initially hydrostatic stress will develop shear stress under isotropic straining. This situation is depicted in Figure 3(b).

4.2. Oedometer test

The lateral deformation is confined in an oedometer test, i.e. $D_{22} = D_{33} = 0$, which is particularly relevant when simulating the earth pressure exerted on earth retaining structures. The derivation of the governing equations and the solution procedure mimic those for isotropic straining and will therefore not repeated.

The numerical results are shown in Figure 4 for a bedding angle of $\theta = 30°$. As may be seen from Figure 4(a) that the non-linear stress–strain behaviour is well reproduced for both loading and unloading. From the stress paths in Figure 4(b) follows that the two lateral stresses are

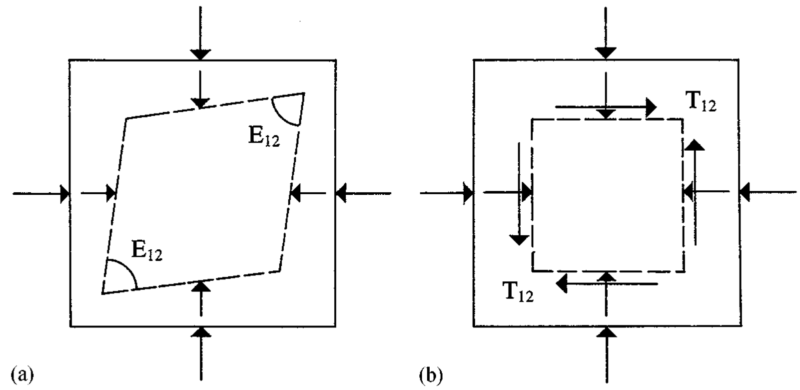

Figure 3. Deformation mode for isotropic stressing (a); stress state for isotropic straining (b)

generally not equal with the in-plane stress T_{22} slightly larger than the out-plane stress T_{33}. As a consequence, there are two distinct coefficients of earth pressure at rest K_0 defined by the ratio of lateral stress to vertical stress. We designate the ratio of in-plane stress to vertical stress by K_{021} and the ratio of out-of-plane stress to vertical stress by K_{031}. In Figure 5 the effect of the bedding angle on the coefficient of earth pressure at rest is shown. We observe that the two coefficients K_{021} and K_{031} are identical for loading normal to the bedding plane. The difference between K_{021} and K_{031} increases with increasing bedding angle and reaches its maximum when loading along the bedding plane. Also the coefficients K_{021} and K_{031} themselves increase from less than 0.4 at $\theta = 0°$ to over 0.5 at $\theta = 90°$.

4.3. Triaxial test

Triaxial tests are most widely used to investigate the deformation and strength behaviour of soils. Usually, mixed boundary conditions are applied with equal lateral stresses $T_{22} = T_{33}$ exerted by cell fluid in a pressure chamber and the vertical straining D_{11} exerted by a loading piston.

The underlying equations can be obtained analogously to isotropic stressing. The expressions are, however, rather lengthy since neither the components of stress nor of strain rate are equal. We omit the concrete expressions by writing out the four non-trivial equations symbolically:

$$\dot{T}_{11} = h_{11}(T_{11}, T_{22}, T_{33}, T_{12}, D_{11}, D_{22}, D_{33}, D_{12}) \tag{36}$$

$$\dot{T}_{22} = h_{22}(T_{11}, T_{22}, T_{33}, T_{12}, D_{11}, D_{22}, D_{33}, D_{12}) \tag{37}$$

$$\dot{T}_{33} = h_{33}(T_{11}, T_{22}, T_{33}, T_{12}, D_{11}, D_{22}, D_{33}, D_{12}) \tag{38}$$

$$\dot{T}_{12} = h_{12}(T_{11}, T_{22}, T_{33}, T_{12}, D_{11}, D_{22}, D_{33}, D_{12}) \tag{39}$$

where h_{ij} are functions of the coefficients of anisotropy and bedding angle as obtained for the specific constitutive equation (28).

Consider a triaxial compression test starting from an initial stress state. Note that the magnitude of strain rate is of unimportance for rate independent constitutive equations.

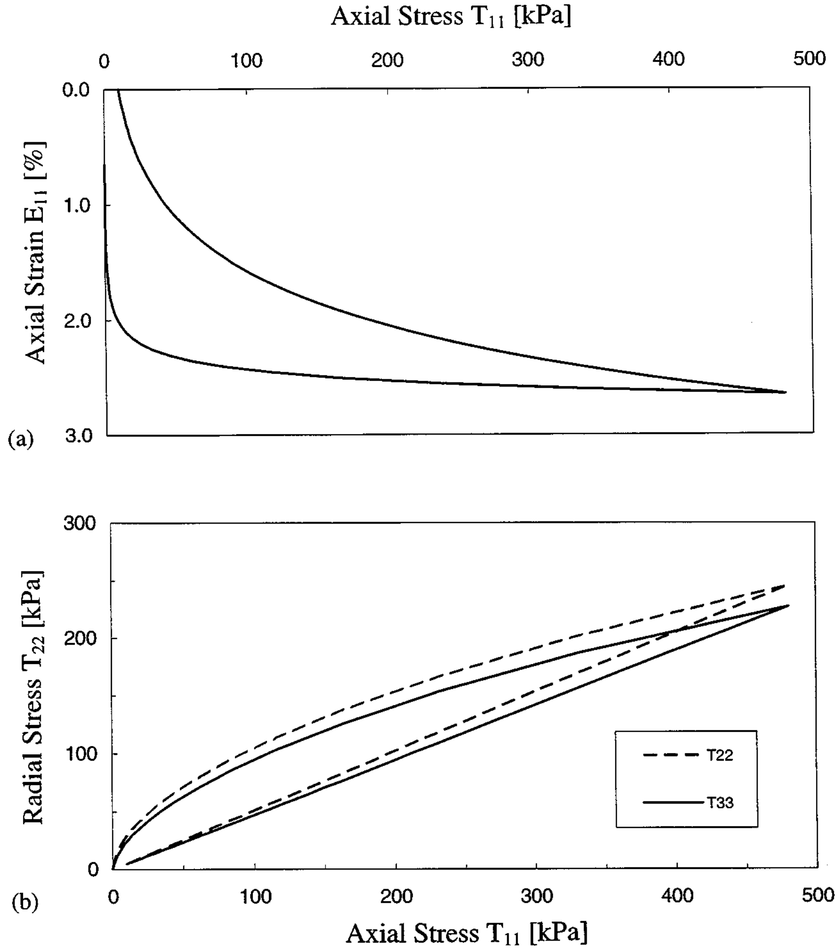

Figure 4. Simulation of oedometer test: (a) stress–strain curve; (b) stress path

Therefore, we may take $D_{11} = +1$ for compression and $D_{11} = -1$ for extension. The boundary conditions in a conventional triaxial test with constant lateral stresses can be written out as follows: $\dot{T}_{22} = 0$, $\dot{T}_{33} = 0$, $\dot{T}_{12} = 0$. Making use of these boundary conditions, the last three equations, (37)–(39), contain three unknown strain rates and can be solved for D_{22}, D_{33} and D_{12}. The strain rates obtained in this way are set into the first equation (36) to get the axial stress rate. The axial stress is then updated and serves as the initial stress for the next calculation.

Numerical simulation of triaxial compression tests with a constant confining pressure of $T_{22} = T_{33} = 100\,\text{kPa}$ is shown in Figure 6. The stress–strain curves in Figure 6(a) show that the tangential stiffness decreases gradually from the beginning of deviatoric loading and vanishes at large strain. This is one of the hallmarks of our constitutive model since there is no elastic domain and the stress–strain behaviour is nonlinear from the very beginning of loading. We further observe that the stiffness and strength depend highly on the bedding angle with their highest

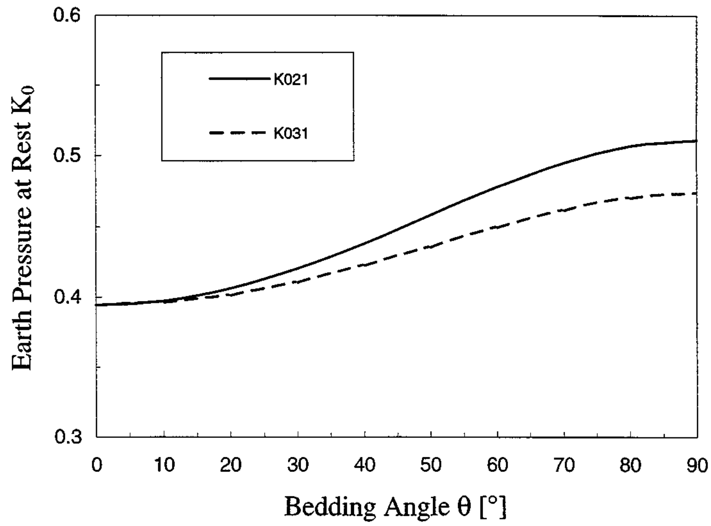

Figure 5. Predicted relation between K_0 and θ

values attained when the axial stress is perpendicular to the bedding plane and their lowest values attained when the axial stress is along the bedding plane. As a pendant to the stress–strain curves, the volume change is shown in Figure 6(b). The dilatancy is seen to become less pronounced with increasing bedding angle. This corresponds fairly well with the observations on the stress–strain curves. These characteristics of the simulation are corroborated qualitatively by numerous laboratory tests.

Figure 7 shows the deformation mode during the test. The results can be compared to the deformation mode of isotropic stressing in Figure 2(a). As can be seen from Figure 7, the two normal strains are nearly equal, while the shear strain develops at a much lower rate than the normal strains. A specimen of right cylinder will be skewed during the test. In a triaxial test with mixed boundary conditions, shear stress will develop along the boundary of strain control. Along the boundary of stress control, however, the shear stress component vanishes. This gives rise to an inhomogeneous stress state within the specimen. In this sense the triaxial test with mixed boundary conditions is not well suited for investigating anisotropic specimens. An alternative will be the so-called directional shear apparatus as reported by Arthur *et al.*,[23] (1977) where all components of the average stress can be controlled.

The flexibility of our constitutive equation in modelling the variation of strength with the bedding angle is demonstrated in Figure 8, where the friction angle at triaxial compression is depicted over the bedding angle. The curves are obtained by varying β while keeping $\alpha = 0{\cdot}90$ and $\gamma = 1{\cdot}10$. It can be seen that a fairly wide range of dependence of the friction angle on the bedding angle can be covered by varying the parameter β. Another interesting feature which can be observed from Figure 8 is that the parameter β has no effect on the friction angle at $\theta = 0^\circ$ and $\theta = 90^\circ$. Further parametric study shows that the parameter α has major effect on the friction angle at $\theta = 0^\circ$ and minor effect on the friction angle at $\theta = 90^\circ$, while the parameter γ has minor effect on the friction angle at $\theta = 0^\circ$ and major effect on the friction angle at $\theta = 90^\circ$. These observations may be used as guidelines when calibrating the coefficients of anisotropy. Compari-

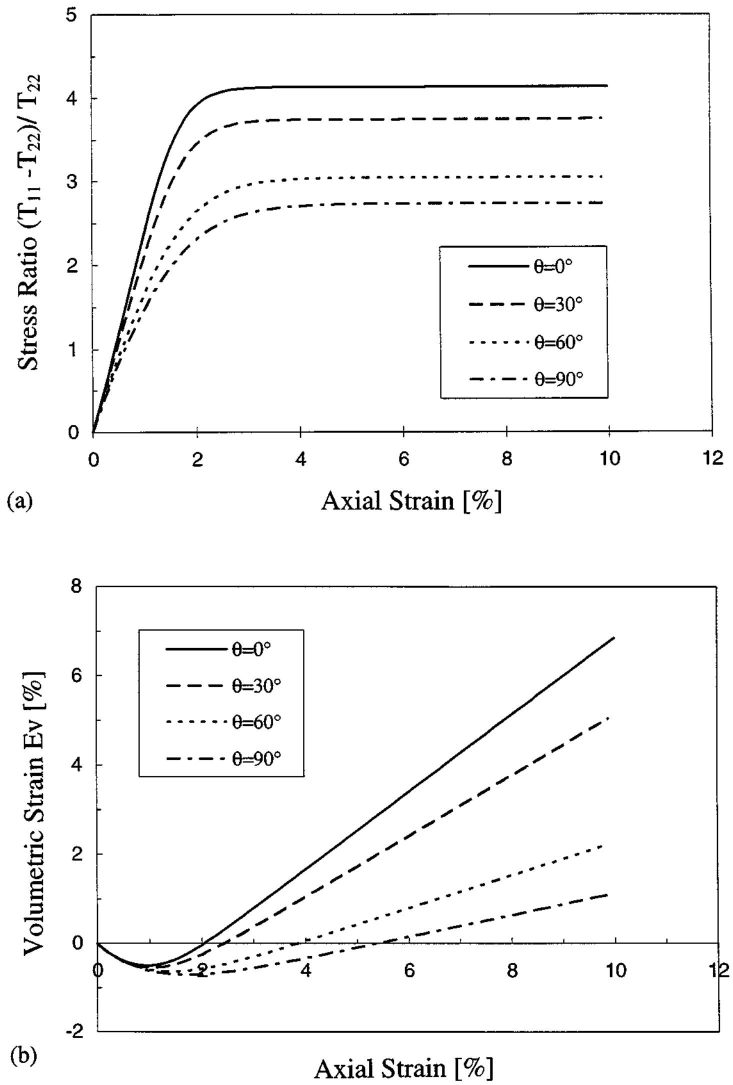

Figure 6. Simulation of triaxial tests with $\sigma_3 = 100$ kPa: (a) stress ratio vs. axial strain; (b) volumetric strain vs. axial strain

son with experimental data is made in Figure 9 by presenting the calculated and the experimentally obtained friction angle in triaxial compression over the bedding angle. The tests conducted by Oda[2] on Toyoura sand and by Arthur and Phillips[1] on Ham river sand are chosen, since they show virtually different dependence of ϕ on θ. While the data of Reference 2 show an initially weak dependence in the range $\theta = 0°$ to $\theta = 30°$ and a subsequently strong dependence in the range $\theta = 30°$ to $\theta = 90°$, the data by Arthur and Phillips[1] show the contrary tendency of an initially strong dependence in the range $\theta = 0°$ to $\theta = 60°$ and a subsequently weak dependence in the range $\theta = 60°$ to $\theta = 90°$. As may be seen from Figure 9, the experimental data are well fitted by adapting the coefficients of anisotropy.

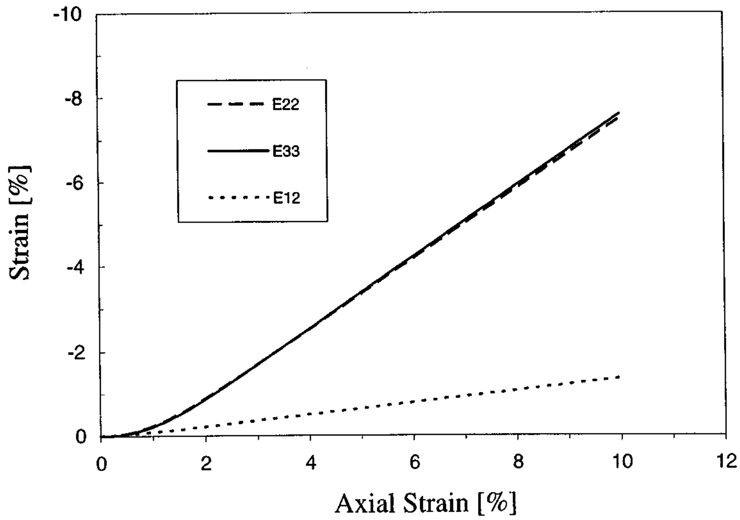

Figure 7. Evolution of strains during a triaxial test

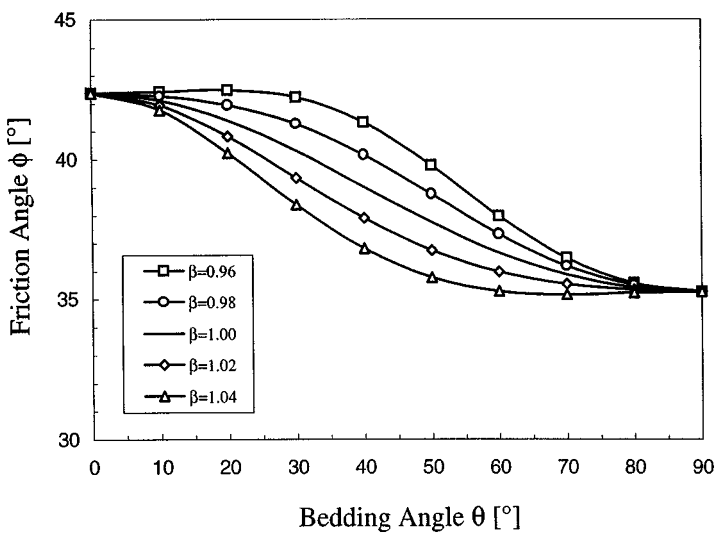

Figure 8. Predicted relations between ϕ and θ

4.4. Failure surface and flow rule

One of the distinct features of our constitutive equation lies in that many concepts such as failure surface and flow rule, which have to be specified *a priori* in plasticity theory, can be obtained *a posteriori* as outcomes of the constitutive model. The relevant aspects for the hypoplastic model of inherent isotropy are treated in the recent work by Wu and Niemunis.[17]

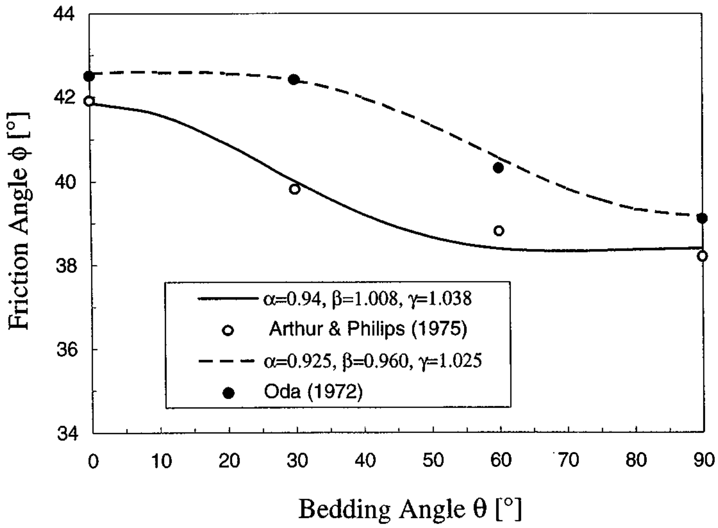

Figure 9. Comparison between numerical and experimental triaxial tests

With reference to constitutive equation (19), we proceed to derive explicit expressions of the failure surface and the flow rule for transverse anisotropy. A material element is said to be at failure, if for a given stress state there exists a nonvanishing strain rate such that the stress rate vanishes

$$\overset{\circ}{\mathbf{T}} = \mathsf{L}(\mathbf{T})\,\mathbf{D} + \mathbf{M}(\mathbf{A}, \mathbf{N})\|\mathbf{D}\| = \mathbf{0} \tag{40}$$

where $\mathsf{L} = \partial\mathbf{L}/\partial\mathbf{D}$ by making use of Euler's theorem for homogeneous functions. The above equation is an isotropic tensorial function of **T**, **D** and **A**. Due to rate independence, equation (40) is necessarily homogeneous of the zeroth degree in **D**. Let us first consider the flow rule at failure, namely the direction of strain rate corresponding to the vanishing stress rate. Since the strain rate in concern is other than zero, equation (40) can be divided by the norm of stretching to give

$$\frac{\mathbf{D}}{\|\mathbf{D}\|} = -\,[\mathsf{L}(\mathbf{T})]^{-1}\mathbf{M}(\mathbf{A}, \mathbf{N}) \tag{41}$$

Note that only the direction of strain rate at failure is specified by (41) and there is no one-to-one correspondence between $\overset{\circ}{\mathbf{T}}$ and **D** at failure. Although the direction of strain rate at failure is an isotropic function of **T** and **A**, the flow rule with respect to stress tensor is, however, anisotropic. As a consequence, **T** and **D** are generally not coaxial. By coaxiality we mean two tensors with the same principal direction. The non-coaxiality between **T** and **D** at failure will be dealt with in detail thereafter.

Constitutive equation (19) of transverse anisotropy is known to reduce to the constitutive equation of inherent isotropy (15) when $\alpha = 1, \beta = 1, \gamma = 1$. In this case, the flow rule (41) reduces to

$$\frac{\mathbf{D}}{\|\mathbf{D}\|} = -\,[\mathsf{L}(\mathbf{T})]^{-1}\mathbf{N}(\mathbf{T}) \tag{42}$$

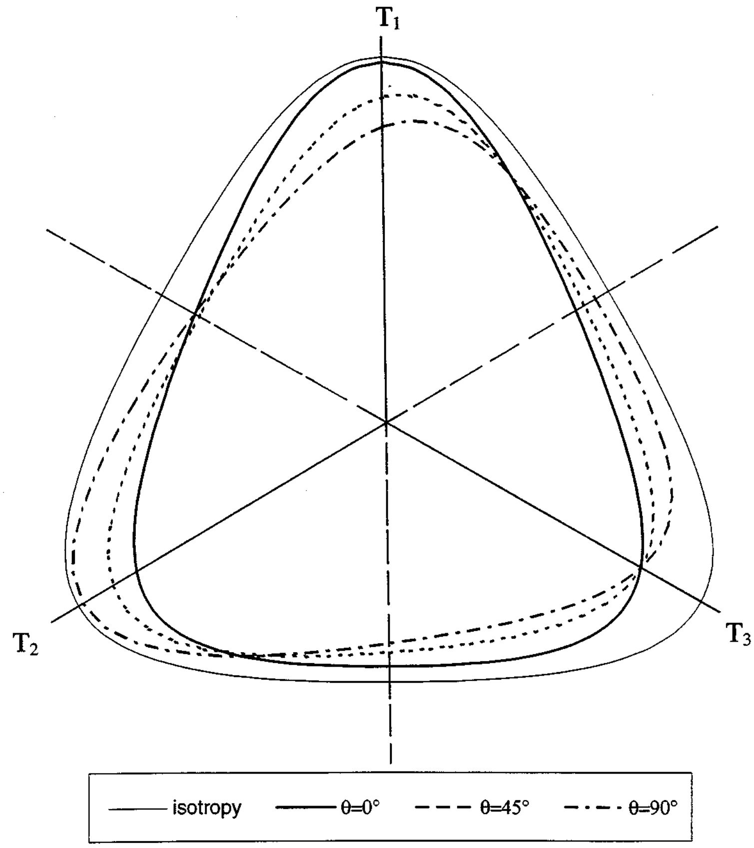

Figure 10. Failure surfaces on a deviatoric plane

Since the direction of strain rate in the above expression is an isotropic function only of stress, the tensors $\mathbf{T}$ and $\mathbf{D}$ are known to be coaxial.

Substitution of (41) into the following definition of the norm of strain rate $\|\mathbf{D}\|$:

$$\frac{\operatorname{tr}\mathbf{D}^2}{\|\mathbf{D}\|^2} = 1 \tag{43}$$

we obtain the expression for the failure surface

$$f(\mathbf{T}, \mathbf{A}) = [\mathbf{M}(\mathbf{A}, \mathbf{N})]^{\mathrm{T}}[[\mathsf{L}(\mathbf{T})]^{-1}]^{\mathrm{T}}[\mathsf{L}(\mathbf{T})]^{-1}\mathbf{M}(\mathbf{A}, \mathbf{N}) - 1 = 0 \tag{44}$$

Again, the above equation is an isotropic function of its arguments $\mathbf{T}$ and $\mathbf{A}$. However, the failure surface is an anisotropic function of $\mathbf{T}$ due to the dependence on $\mathbf{A}$. For the case $\alpha = 1$, $\beta = 1$, $\gamma = 1$, the expression for the failure surface of inherently isotropy is recovered.[17] From the above derivations it may be seen that failure concerns two aspects, namely kinematical and dynamical. Consequently, there are two equations which follow from the definition of failure. The first one specifies the direction of strain rate at failure and is called the flow rule, while the second one

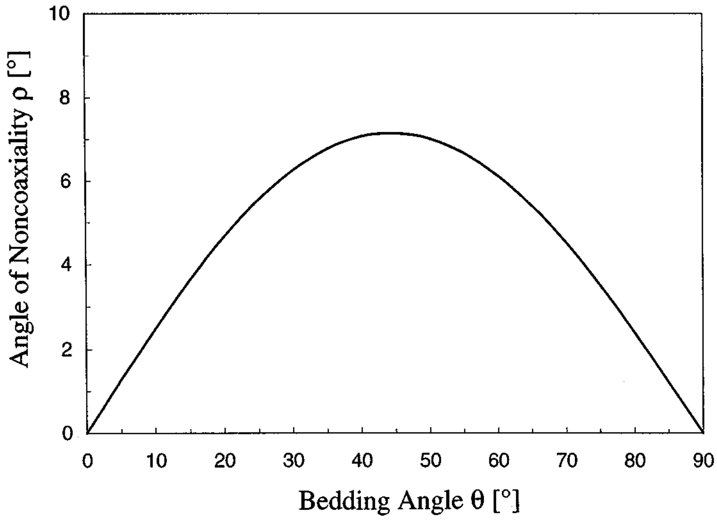

Figure 11. Predicted relation between noncoaxiality and bedding angle

concerns the stress at failure and is termed the failure surface. A further remark is relevant to whether the flow rule (41), with reference to the failure surface (44), is associated. A perusal of (41) and (44) suggests that the flow rule is nonassociated, since in general $\partial f(\mathbf{T})/\partial \mathbf{T} \neq -\mathsf{L}^{-1}\mathbf{M}$. Figure 10 shows the failure surfaces calculated from the extended constitutive equation for three bedding angles together with the failure surface of inherent isotropy. As might be expected, the failure surface of inherent isotropy possesses three orthogonal axes of symmetry, whereas the failure surfaces of inherent anisotropy possess only one axis of symmetry.

Finally, we turn our attention to non-coaxiality between **T** and **D** at failure. The assumption of coaxiality between stress and plastic strain rate in plasticity theory has been a controversial issue in several experimental studies indicating certain degree of non-coaxiality between **T** and **D**.[23, 24] We will not attempt to model the experimental observations quantitatively. Rather, we will show that non-coaxiality can be obtained with the extended constitutive model. For the sake of simplicity, we will consider a triaxial compression test. With reference to the coordinate system x' the non-coaxiality between **T** and **D** can be measured by the angle of the major principal strain rate at failure defined by $\rho = 0.5 \arctan\,[D_{12}/(D_{11} - D_{22})]$. The results presented in Figure 11 show that the stress and strain rate at failure are coaxial only when loaded perpendicular to or along the bedding plane. We have otherwise non-coaxiality with the highest deviation from coaxiality of about 8° at the bedding angle of $\theta = 45^\circ$.

5. CONCLUDING REMARKS

A constitutive model for transversely anisotropic sand is presented, which is based exclusively on the methodology of rational mechanics. One of the hallmarks of our model lies in the fair balance between the elegant formulation and the predictive capability. This is achieved by treating the strain rate as a whole rather than the separation of the strain rate into elastic and plastic parts as is the case in plasticity theory. The model is further characterized by the predictive nature of some

well-established concepts, e.g. failure criterion and flow rule, which must be specified beforehand in plasticity theory.

The model can be readily extended to account for the effect of the stress level and the void ratio on anisotropy. The pendant for the case of inherent isotropy has been presented by Wu *et al.*,[18] where the void ratio and the critical state have been included. The test results in the literature indicate that loose sand seems to show more pronounced anisotropy than dense sand. In fact, the void ratio as a scalar may be regarded as the first degree approximation to the complex structure of granular materials. A more realistic description may be achieved by invoking a so-called fabric tensor.

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